CVXOPT: Convex optimization with Python

Joachim Dahl
Department of Communication Technology
Aalborg University

Lieven Vandenberghe
Department of Electrical Engineering
University of California, Los Angeles

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Outline

• Motivation

• Matrix library

• Optimization solvers

• Applications and modeling
Convex optimization

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

\(x = (x_1, \ldots, x_n)\) is the optimization variable

\(f_0, \ldots, f_m\) are convex functions

- Includes least-squares, linear programming, many other problem classes.
- Convex optimization problems are fundamentally tractable.
History

- 1940s: linear programming

\[
\text{minimize} \quad c^T x \\
\text{subject to} \quad a_i^T x \leq b_i, \quad i = 1, \ldots, m
\]

- 1950s: quadratic programming

- 1960s: geometric programming

- 1990s: semidefinite programming, second-order cone programming, quadratically constrained quadratic programming, robust optimization, sum-of-squares programming, \ldots
New applications since 1990

• Linear matrix inequality techniques in control

• Circuit design via geometric programming

• Support vector machine learning via quadratic programming

• Semidefinite programming relaxations in combinatorial optimization

• Applications in statistics and machine learning, signal processing, communications, image processing, quantum information theory, finance, structural optimization, . . .
Interior-point methods

Linear programming

- 1984 (Karmarkar): first practical polynomial-time algorithm
- 1984-1990: efficient implementations for large-scale LPs

Nonlinear convex optimization

- Since 1990: extensions and high-quality software packages
Embedded convex optimization

- Efficient, in theory and practice
- Performance is similar across wide range of problem dimensions, problem data, problem classes
- Result is independent of choice of starting point
- Detect infeasibility
- Controlled by a small number of easily tuned algorithm parameters

Hence, useful for embedded optimization, i.e., integrated in applications.
Software for convex optimization

- Proprietary packages: CPLEX, MOSEK, . . .
- Free, based on Matlab: SeDuMi, SDPT3, YALMIP, CVX, . . .
- Free, in the form of C libraries: GLPK, DSDP, SDPA, . . .
- Exploit certain types of problem structure (sparsity).

In practice, integrating convex optimization in applications often requires:

- substantial coding effort for interfacing with application,
- proprietary software,
- implementations that exploit (non-sparse) problem structure.
CVXOPT

Goals

• Free, portable package for convex optimization.
• Simplify development of embedded convex optimization applications.
• Take advantage of Python’s extension modules.
• Matrix library for prototyping new algorithms.

Not: general-purpose computing package like Matlab, Octave
Current release (0.8.1)

• Dense and sparse matrices

• Interfaces to BLAS, LAPACK, CHOLMOD, UMFPACK, FFTW.

• Python solver for linear and semidefinite programming

• Python solver for nonlinear convex optimization

• Interfaces to solvers in GLPK, MOSEK and DSDP

• Modeling for piecewise-linear convex optimization problems
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Dense and sparse matrices

Dense matrices

• Three types: integer, double, complex
• Two-dimensional arrays stored in column-major order
• Compatible with NumPy via Array Interface

Sparse matrices

• Two types: double, complex
• Compressed column storage format
Matrix arithmetic

Overloaded arithmetic: \( A + B, \ A - B, \ A * B, \ . . . \)

In-place addition and scalar multiplication: \( A += B, \ A *= c, \ . . . \)
- Allowed if the dimensions and type of \( A \) do not change
- Modify existing matrix object (\( A += B \) is not the same as \( A = A + B \))

Indexing: with one or two arguments, using index sets or Python slices

Matrix functions: elementwise multiplication, cosine, exp, . . .
Basic Linear Algebra Subprograms (BLAS) interface

- Larger set of arithmetic operations (e.g., \( A := \alpha A + \beta xy^T, \ldots \))
- Separate routines for symmetric, triangular, banded matrices
- Always work ‘in-place’, i.e., without creating new matrices
- Some functions overloaded to allow sparse arguments
LAPACK interface

- LU, Cholesky, LDL^T, QR factorization
- Solution of dense linear equations
- Linear least-squares and least-norm problems
- Symmetric and Hermitian eigenvalue decomposition
- Singular value decomposition
Sparse linear equations

- Interface to UMFPACK (LU factorization) and CHOLMOD (Cholesky)
- Separate functions for symbolic and numeric factorization

Example: compute

\[ A^{-T} B^{-1} A^{-1} b \]

where \( A, B \) have same sparsity pattern

Fs = umfpack.symbolic(A)
FA = umfpack.numeric(A, Fs)
FB = umfpack.numeric(B, Fs)
umfpack.solve(A, FA, b)
umfpack.solve(B, FB, b)
umfpack.solve(A, FA, b, trans='T')
Differences with Matlab and Octave

- Only a small subset of Matlab/Octave functionality
- Direct interface to BLAS, LAPACK, CHOLMOD, UMFPACK
- Function arguments are passed by reference
- Python scoping rules are different

**Example:** solve positive definite $Ax = b$ with different righthand sides

```python
def solver(A):
    lapack.potrf(A)  # A = L*L^T; A := L
    def f(x):
        lapack.potrs(A,x)  # x := L^{-T}*L^{-1}*x
    return f

f = solver(A)
f(x)  # x := A^{-1}*x
f(y)  # y := A^{-1}*y
```
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& \quad Ax = b
\end{align*}
\]

- Simple solver: user provides functions for evaluating constraint functions and derivatives
- Expert solver: user also provides function for solving KKT systems
- Interfaces for quadratic programming and geometric programming
Example (simple solver): logistic regression

minimize \quad f(x) = c^T x + \sum_{i=1}^{m} \log(1 + \exp((Ax)_i))

m, n = A.size
def F(x=None, z=None):
    if x is None: return 0, matrix(0.0, (n,1))
w = exp(A*x)
    # function value and gradient
    f = c.T*x + sum(log(1+w))
    grad = c + A.T * div(w, 1+w)
    if z is None: return f, grad.T
    # Hessian
    H = A.T * spmatrix(div(w,(1+w)**2), range(m), range(m)) * A
    return f, grad.T, z[0]*H

sol = solvers.cp(F)
Example (expert solver): 1-norm regularized regression

\[
\text{minimize} \quad \|Ax - y\|_2^2 + \|x\|_1 \quad (A \in \mathbb{R}^{m \times n}, n \gg m)
\]

- Equivalent to a quadratic program.

- Every iteration involves solving a set of linear (KKT) equations:

\[
\begin{bmatrix}
\lambda A^T A & 0 & I & -I \\
0 & 0 & -I & -I \\
I & -I & -D_1^{-1} & 0 \\
-I & -I & 0 & -D_2^{-1}
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta y \\
\Delta z_1 \\
\Delta z_2
\end{bmatrix}
= b
\]

Reduces to a system of the form \((ADA^T + I)v = r\).

- Structure can be exploited by providing a routine for KKT system.
Linear cone programming

minimize \( c^T x \)
subject to \( Gx \leq h \)
\( Ax = b \)

Inequalities include vector (componentwise) inequalities and matrix inequalities.

- Simple solvers: use standard matrix format
- Expert solver: user provides function for solving KKT system
Example: 1-norm approximation

\[
\begin{aligned}
\text{minimize } & \| Pu - q \|_1 \\
\text{Equivalent to the LP } & \\
\text{minimize } & 1^T y \\
\text{subject to } & -y \leq Pu - q \leq y
\end{aligned}
\]

Every iteration involves solving sets of linear equations:

\[
\begin{bmatrix}
0 & 0 & P^T & -P^T \\
0 & 0 & -I & -I \\
P & -I & -D_1^{-1} & 0 \\
-P & -I & 0 & -D_2^{-1}
\end{bmatrix}
\begin{bmatrix}
\Delta u \\
\Delta y \\
\Delta z_1 \\
\Delta z_2
\end{bmatrix} = b.
\]

Reduces to a system of the form

\[
P^T DP \Delta u = r
\]
def l1(P, q):
    m, n = P.size
    def Fi(x, y, alpha=1.0, beta=0.0, trans='N'):
        if trans=='N':
            ... # evaluate y := alpha*[P, -I; -P, -I]*x + beta*y
        else:
            ... # evaluate transpose
    def kktsolver(di):  # Return a solver for KKT system
        # Compute and factor A = P'*D*P
        ...
        def f(x,y,z):  # Solve KKT system using factorization
            ...
        return f

    c = matrix(size=(m+n,1));  c[:n], c[n:] = 0.0, 1.0
    h = matrix(size=(2*m,1));  h[:m], h[m:] = q, -q
    sol = solvers.conelp(c, kktsolver, Gl=Fi, hl=h)
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Applications

- Interactive shell
- Spreadsheet interfaces (gnumeric)
- Database applications
- Networking and distributed applications
- Graphical user interfaces
- Web interfaces
- Modeling via built-in Python interpreter
Piecewise-linear optimization

- Python object for (vector) variables
  
  ```python
  x = variable(5, 'x')
  ```

- Create PWL functions by overloaded arithmetic operations, `max()`, `min()`, `abs()`, `sum()`, indexing and slicing
  
  ```python
  f = sum(abs(A*x+b))
  ```

- Create constraints by overloaded comparison operators
  
  ```python
  c1 = (f <= 1)
  c2 = (sum(x) == 1)
  ```

- Define optimization problem by specifying objective and constraints
  
  ```python
  prob = op(x[1], [c1, c2])
  ```
Example: Penalty approximation

**Chebyshev norm:** minimize $\|Ax - b\|_\infty$

**1-norm:** minimize $\|Ax - b\|_1$

**Piecewise-linear penalty:** minimize $\sum_k \phi((Ax - b)_k)$

Can be expressed as LPs, by introducing auxiliary variables and constraints.
Example: distribution of residuals $Ax - b \ (A \in \mathbb{R}^{500 \times 200})$. 
1. Minimize $\|Ax - b\|_\infty$:

   \[
   x_1 = \text{variable}(n) \\
   \text{prob1} = \text{op}(\max(\text{abs}(A*x_1-b))) \\
   \text{prob1}.\text{solve}()
   \]

2. Minimize $\|Ax - b\|_1$:

   \[
   x_2 = \text{variable}(n) \\
   \text{prob2} = \text{op}(\text{sum}(\text{abs}(A*x_2-b))) \\
   \text{prob2}.\text{solve}()
   \]

3. Minimize $\sum_k \phi((Ax - b)_k)$:

   \[
   x_3 = \text{variable}(n) \\
   \text{prob3} = \text{op}(\text{sum}(\max(0, \text{abs}(A*x_3-b)-0.75, 2*\text{abs}(A*x_3-b)-2.25))) \\
   \text{prob3}.\text{solve}()
   \]
Summary

Applications

• Embedded convex optimization via Python’s extension libraries
• In interactive mode, a user-friendly package for convex optimization
• Prototyping of numerical algorithms (linear algebra and optimization)

Some future plans

• Iterative linear algebra
• Second-order cone programming
• Python implementation of CVX modeling tool (Grant, Boyd, Ye)

www.ee.ucla.edu/~vandenbe/cvxopt